THE INCREASE OF SUMS AND PRODUCTS DEPENDENT ON $(y_1, \cdots, y_n)$ BY REARRANGEMENT OF THIS SET

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ABSTRACT

Let $F(u, v)$ be a symmetric real function defined for $\alpha < u, v < \beta$ and assume that $G(u, v, w) = F(u, v) + F(u, w) - F(v, w)$ is decreasing in v and w for $u \leqq \min(u, v)$. For any set $(y) = (y_1, \ldots, y_n)$, $\alpha < y_i < \beta$, given except in arrangement $\sum_{i=1}^{n} F(y_i, y_{i+1})$ where $y_{n+1} = y_1)$ is maximal if (and under some additional assumptions only if) (y) is arranged in circular symmetrical order. Examples are given and an additional result is proved on the product $\Pi_{i=1}^{n}[(y_{2i-1}y_{2i})^m + a_1(y_{2i-1}y_{2i})^{m-1} + \ldots + a_m]$ where $a_k \geqq 0$ and where the set $(y) = (y_1, \ldots, y_n)$, $y_i \geqq 0$ is given except in arrangement. The problems considered here arose in connection with a theorem by A. Lehman [1] and a lemma of Duffin and Schaeffer [2].

We start with some definitions given in [1].

The sets $(y^-) = (y_1^-, y_2^-, \cdots, y_n^-)$ and $(^-y) = (^-y_1, \cdots, ^-y_n)$ are *symmetrically decreasing rearrangements of* an ordered set $(y) = (y_1, \cdots, y_n)$ of n real numbers if

(1) $$y_1^- \leqq y_n^- \leqq y_2^- \leqq \cdots \leqq y^-_{[(n+2)/2]}$$

and

(2) $$^-y_n \leqq {^-y_1} \leqq {^-y_{n-1}} \leqq \cdots \leqq {^-y_{[(n+1)/2]}}.$$

A circular rearrangement of an ordered set $(y) = (y_1, \cdots, y_n)$ is a cyclic rearrangement of (y) or a cyclic rearrangement followed by inversion.

An ordered set $(y) = (y_1, \cdots, y_n)$ of n real numbers is arranged in *circular symmetrical order* if one of its circular rearrangements is symmetrically decreasing. It follows that the sets (y^-), (^-y) are arranged in circular symmetrical order and so is the set $(y) = (y_1, \cdots, y_n)$ if either

(3) $$y_1 \leqq y_2 \leqq y_n \leqq y_3 \leqq y_{n-1} \leqq \cdots \leqq y_{[(n+3)/2]}$$

or

(4) $$y_2 \leqq y_1 \leqq y_3 \leqq y_n \leqq y_{n-1} \leqq \cdots \leqq y_{[(n+4)/2]}$$

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holds.

THEOREM 1. *Let $F(u,v)$ be a symmetric real function defined for*

$$\alpha < u, v < \beta \qquad -\infty \leqq \alpha < \beta \leqq \infty,$$

and assume that the function

$$G(u,v,w) = F(u\ v) + F(u,w) - F(v,w), \qquad \alpha < u,v,w < \beta \tag{5}$$

is decreasing in v and w for $u \leqq \min(v,w)$.

Let the set $(y) = (y_1,\cdots,y_n)$ $\alpha < y_i < \beta$, $i = 1,\cdots, n$ be given except in arrangement. Then

$$S_n = \sum_{i=1}^{n} F(y_i, y_{i+1}), \quad (y_{n+1} = y_1) \tag{6}$$

is maximal if (y) is arranged in circular symmetrical order.

Moreover, if $G(u\ v,w)$ is strictly decreasing in v and w for $u < \min(v,w)$ and

$$F(u,u) = G(u,v,u) = G(u,u,w) > G(u,v,w) \qquad u < \min(v,w) \tag{7}$$

and if, in addition, no three elements of (y) have the same value, then (6) attains its maximum only if (y) is arranged in circular symmetrical order.

Proof. As the proof is similar to the proof in [1], we give here only a short outline.

The first assertion of the theorem is equivalent to

$$\begin{gathered} S_n^- = \sum_{i=1}^{n} F(y_i^-, y_{i+1}^-) \geqq \sum_{i=1}^{n} F(y_i, y_{i+1}) = S_n \\ y_{n+1} = y_1, y_{n+1}^- = y_1^- \quad y_1 \leqq y_i \qquad i = 2,\cdots,n. \end{gathered} \tag{8}$$

(8) is proved by induction, using the equalities

$$\begin{gathered} \sum_{i=1}^{n} F(y_i^-, y_{i+1}^-) = \sum_{i=1}^{n-1} F(x_i', x_{i+1}') + G(y_1, x_i', x_{n-1}') \\ \sum_{i=1}^{n} F(y_i, y_{i+1}) = \sum_{i=1}^{n-1} F(x_i\ x_{i+1}) + G(y_1, x_1, x_{n-1}) \\ y_{n+1} = y_1 \quad x_n = x_1, \quad y_{n+1}^- = y_1^-, \quad x_n' = x_1' \end{gathered} \tag{9}$$

where $x_i = y_{i+1}$, $x_i' = y_{i+1}^-$, $i = 1,\cdots,n-1$ (hence $(x') = (^-x)$).

The second assertion is also proved by induction and (9) again allows us to proceed from $n-1$ to n.

For the sets (1,2,3,4,2,2) and (1,2,2,3,4,2) and any symmetric function F the sums (6) are equal; hence it is necessary for the second part of the theorem to assume that no three elements of (y) have the same value.

The theorem yields the following result concerning linear rearrangements

COROLLARY. *Let all the assumptions about $F(u,v)$ and $G(u,v,w)$ of the first assertion of Theorem 1 hold, and assume $\alpha < 0$ $(\alpha < u,v < \beta.)$ If $F(0,v) = F(u,0) = 0$ and if $(x) = (x_1,\cdots,x_n)$ is a set of n positive numbers $0 < x_i < \beta$ $i = 1 \cdots,n$ given except in arrangement then*

$$\sum_{i=1}^{n-1} F(x_i, x_{i+1}) \tag{10}$$

is maximal if (x) is arranged in symmetrical decreasing order.

Proof. Define

$$\begin{aligned} y_{i+1} &= x_i \quad & i &= 1,\cdots,n, \\ y'_{i+1} &= {}^{-}x_i \quad & i &= 1,\cdots,n, \\ y_1 &= y'_1 = 0. \end{aligned} \tag{11}$$

note that $(y') = (y^-)$.
Using (11) the assertion of the corollary is equivalent to

$$\sum_{i=1}^{n+1} F(y'_i, y'_{i+1}) \geqq \sum_{i=1}^{n+1} F(y_i, y_{i+1}) \tag{10'}$$

because $y_1 = 0$ and $F(0,u) = 0$.
But this is (8) because $(y') = (y^-)$, and thus the proof of the corollary is complete.

It can be shown that if in the corollary we assume that the stronger assumptions of the second part of Theorem 1 hold, then (10) attains its maximum only if (x) is symmetrically decreasing.

We give now examples of functions $F(u\ v)$ satisfying the assumptions of Theorem 1.

Define $F(u,v) = f(|u-v|)$, where $f(x)$ is concave decreasing function for $x \geqq 0$. This is a symmetric function of u and v and

$$G(u,v,w) = f(|u-v|) + f(|u-w)| - f(|v-w|)$$

has all the properties which are needed for the first assertion of Theorem 1. If the concavity of $f(x)$ is strict, then $F(u,v) = f(|u-v|)$ and $G(u,v,w)$ has all the properties of Theorem 1. In this special case Theorem 1 becomes Lehman's theorem.

Another class of functions satisfying the first part of Theorem 1 consists of the symmetric functions $F(u,v)$ for which $\partial^2 F/\partial u \partial v \geqq 0$ holds.

Examples of such functions are

$$F(u,v) = u \cdot v,$$

$$F(u,v) = u^s v^r + v^s u^r \qquad s,r > 0 \qquad u,v > 0,$$

$$F(u,v) = \log(u \cdot v + t) \qquad t > \max(0, -u \cdot v).$$

It can be shown for these examples that the assumptions on $F(u,v)$ of both parts of Theorem 1 hold.

From the last example it follows that

$$(12) \qquad \prod_{i=1}^{n} (y_i y_{i+1} + t),\ y_{n+1} = y_1 \ \ t > \max(0, -y_i y_j;\ i \neq j \ \ i = 1,\cdots,n)$$

is maximal when (y) is arranged in circular symmetrical order, and if no three elements of (y) have the same value, then the maximum is attained only if (y) is arranged in circular symmetrical order.

We now turn to another result concerning a product which is a generalization of the following lemma of Duffin and Schaeffer.

LEMMA [2, p 522]. *Let the set $(y) > 0$ of $2n$ nonnegative numbers be given except in arrangement. Then*

$$(13) \qquad \prod_{i=1}^{n} (y_{2i-1} \cdot y_{2i} + t) \qquad t \geqq 0$$

is maximal when (y) is arranged in decreasing order.

Generalizing this result we obtain the following theorem.

THEOREM 2. *Let $(a) = (a_1,\cdots,a_m)$ and $(y) = (y_1,\cdots,y_{2n})$ be sets of nonnegative qumbers where (y) is given except in arrangement, then the product*

$$(14) \qquad \prod_{i=1}^{n} [(y_{2i-1}y_{2i})^m + a_1(y_{2i-1}y_{2i})^{m-1} + \cdots + a_m]$$

attains its maximum when (y) is arranged in decreasing order.

Proof. The proof is by induction on m. $m = 1$ is the lemma of Duffin and Schaeffer. Let y_1 be the maximal term in (y). If between y_1 and y_2 there is a term, let us call it y_3, $y_2 < y_3 < y_1$, we interchange y_2 with y_3 and consider the difference

$$\begin{aligned} A &= [(y_1y_3)^m + a_1(y_1y_3)^{m-1} + \cdots + a_m]\,[(y_2y_4)^m + a_1(y_> y_4)^{m-1} + \cdots + a_m] \\ &- [(y_1y_2)^m + a_1(y_1y_2)^{m-1} + \cdots a_m]\,[(y_3y_4)^m + a_1(y_3y_4)^{m-1} + \cdots + a_m] \\ &= \{[a_1(y_1y_3)^{m-1} + \cdots + a_m]\,[a_1(y_2y_4)^{m-1} + \cdots + a_m] \\ &- [a_1(y_1y_2)^{m-1} + \cdots + a_m]\,[a_1(y_3y_4)^{m-1} + \cdots + a_m]\} \\ &+ \{(y_1y_3)^m[a_1(y_2y_4)^{m-1} + \cdots + a_m] + (y_2y_4)^m[a_1(y_1y_3)^{m-1} + \cdots + a_m] \\ &- (y_1y_2)^m[a_1(y_3y_4)^{m-1} + \cdots + a_m] - (y_3y_4)^m[a_1(y_1y_2)^m + \cdots + a_m]\}. \end{aligned}$$

Let us look at the terms after the equality sign: By the assumption of induction the term in the first braces is nonnegative. The term in the second braces is also nonnegative because it is equal to

$$\sum_{k=1}^{m} a_k(y_1y_2y_3y_4)^{m-k}\,[(y_1^k - y_4^k)\,(y_3^k - y_2^k)]$$

and we assumed that $y_3 > y_2$, $y_1 > y_4$. Therefore $A \geqq 0$ and we can rearrange the set (y) in such a way that two greatest numbers of (y) will appear in the same term of the product (14) without diminishing it. We continue the same process for the remaining terms of the product (14). This completes the proof of the theorem.

The proof shows that if $(y) > 0$ and at least one of the a_k, $k = 1, \cdots, n$, is positive, then the maximum is attained only in those cases in which neither

$$y_{2i-1} < y_k < y_{2i}$$

nor

$$y_{2i} < y_k < y_{2i-1}$$

holds.

References

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